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ANALOG COMPUTERS APPLIED TO ELASTIC-PLASTIC SYSTEMS

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ANALOG COMPUTERS APPLIED TO ELASTIC - PLASTIC SYSTEMS

Leo Schenker[1], A.M. ASCE and Gunther Martin[2]

ABSTRACT

A circuit is described by the use of which differential analog computers can be adapted to solve the equations of motion of systems incorporating springs with elastic-plastic resistance characteristics. The application of the method to a single-story structure is described and illustrated. The circuit can be used in conjunction with multi-degree-of-freedom systems with or without viscous damping. Interesting possibilities for further research are briefly discussed.

INTRODUCTION

The use of differential analog computers for the analysis of the dynamic behavior of elastic systems has become a standard procedure and is fully described in literature.[3] Briefly, the fundamental units of the computer are operational amplifiers which can perform the processes of

1) multiplication by a constant
2) integration.

It is also a characteristic of these units that they will change the sign of the input in addition to carrying out the operation (1) or (2) above. For example, a steady voltage of +10 volts fed into a unit set up to multiply by 2 will be transformed into a steady voltage of -20 volts. A steady positive voltage fed into an integrator will emerge as a voltage increasing at a constant rate, the sign being negative. Thus a combination of operational amplifiers can be so connected as to carry out operations prescribed by a mathematical equation.

As an illustration, the equation of motion of an undamped single-degree-of-freedom system, acted upon by a force F(t), is considered:

$$m\ddot{x} + kx = F(t) \qquad (1.a)$$

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3. "Electronic Analog Computers" Korn & Korn, McGraw-Hill Book Company, Inc., 1952.

The circuit is shown in Fig. 1. To examine this, let it be assumed that the output of the second operational amplifier is -x, as shown in the figure. Feeding this into a potentiometer set to a value k and then into an operational unit to multiply by -1 results in a value kx. Now, F(t) can be represented by a voltage signal similar to the actual force. But equation (1.a) says that $kx - F(t) = -m\ddot{x}$. Therefore, passing kx-F(t) through an integrator, set to multiply by $\frac{1}{m}$ at the same time, should produce $\dot{x}$ and sending $\dot{x}$ through another integrator should result in the output -x. Hence the original assumption that the output of the second unit in Fig. 1 is -x is justified. The variation of x with time can be recorded and represents the solution of (1.a), to a scale which can be determined.

The purpose of the study described here is to develop a procedure suitable for the analysis of systems in which the resistance R(x) to a displacement x is not at all times proportional to x. Specifically, it is desired to solve the equation

$$m\ddot{x} + R(x) = F(t) \qquad (1.b)$$

where R(x) is the elastic-plastic resistance curve shown in Fig. 2(b). In terms of computer circuits, this requires the replacement of the components shown in the dashed box in Fig. 1 by a circuit in which the input -x(t) will produce an output R(x). Once such a circuit is developed, it should be possible to solve (1.b) by the arrangement shown in Fig. 3, in which the R(x)-computer is shown as a solid box. Since information on the sign of $\dot{x}$ will clearly be required, a dotted line supplying this information is also shown. The chief obstacle is that it cannot be prescribed in advance when x will reach a peak value.

Circuit for the Elastic-Plastic Resistance Function

First it is necessary to define R(x) for any value of x. R(x) is not a single valued function: it depends on the history of x as well as on x. By reference to Fig. 2(b) three states can be defined. This is done in Table 1 below.

Table I

State	x	History of x	R(x)
1.	$x < x_L$	x never before has reached x_L	$R(x) = kx$
2.	$x < x_L + \frac{b}{k}$	x has been $x_L + \frac{b}{k}$	$R(x) = kx - b$
3.	$x > x_L$	x has never been larger than now	$R(x) = R_L = kx_L$

A circuit which will follow these instructions and will therefore compute R(x) in accordance with Fig. 2(b) is shown in Fig. 4. Fig. 5 is a photograph of an oscillograph record showing the relation between R(x) and x in parametric form, since they are both necessarily plotted against time. Before describing the salient features of the circuit, it should be pointed out that if it were desired to reproduce the relation between R(x) and x for increasing x alone ($\dot{x}$ never negative), then only that part of the circuit in Fig. 4 above the dash-dot line would be required. If only this

upper portion of the circuit were used, then for decreasing values of x (when $x > x_L$) R(x) would retrace BA (Fig. 2-b) rather than the line BC given by the equation $R(x) = kx - b$. It will be observed from Fig. 5, that as soon as x decreases ($x > x_L$), R(x) decreases also, i.e. R(x) is generated in accordance with the instructions in the table.

In the circuit shown in Fig. 4 the output of amplifier A1 is kx, corresponding to OAD in Fig. 2(b). Since diode D2 is biased to pass any positive voltage in excess of R_L, the output of amplifier A2 follows lines OAB in Fig. 2(b), with a negative sign (states 1 and 3). The difference between the outputs of A1 and A2 is b (Fig. 2-b). Amplifier A4 may therefore be called the b- computer.

Amplifier A5 acts as a sign changer, except that condensers are used instead of the usual resistors. (The 1000 ohm resistor between relay S1 and A5 is included for circuit stability). Relay S1 is normally open and is closed momentarily at the instant that $\dot{x}$ becomes negative. Thus A5 is used as a memory for the last value of b and it is made to store this value during state 2, when $R(x) = kx-b$, that is BCB in Fig. 2-b. The manner in which R(x) is made to follow BCB is this: as soon as $\dot{x}$ becomes negative, relay S1 closes and b is stored on A5. A split second later S3 closes allowing b to be an additional input into A1. The total input into A1 is now -kx+b. Its output is kx-b.

As R(x) comes up CB (Fig. 2-b) and reaches the value of R_L, relay S3 is opened and the input of b into A1 is interrupted. Thus the output of A1 jumps back to kx; the output of A2 is limited to $-R_L$ as before (state 3). This can be repeated as often as necessary.

The rest of the circuit does not require detailed description. Examination will reveal that the positions of the relays are as follows:

S1 - Normally open; closed momentarily only when $\dot{x}$ becomes negative.

S2 - Spring loaded closed; open when $\dot{x}$ is negative.

S3 - Spring loaded closed; open when $\dot{x}$ is positive. and $R(x) = R_L$.

S4 - Spring load in position a; in position b, allowing condenser to discharge through 100,000 ohms resistor, when $\dot{x}$ is positive.

Application to a Single-Degree-of-Freedom System

If the circuit shown in Fig. 4 is put into the place of the solid box in Fig. 3, it should be possible to solve (1.b), the equation of motion for a single-degree-of-freedom system with an elastic-plastic spring. To confirm whether this can in fact be done, a numerical example was solved which has been used by Professor N. M. Newmark[4]. This might represent a single-story structure acted upon by a variable force as shown in Fig. 6. After dividing by the numerical value of the mass, the equation of motion in inch-kip-second units is

4. "Methods of Analysis for Structures Subjected to Dynamic Loading", Report to Physical Vulnerability Branch, Air Targets Division, Directorate of Intelligence, U. S. Air Force, Revised December 1950.

$$\ddot{x} + R(x) = 3720t - 7300t^2 \qquad (2)$$

$$0 < t < 0.51$$

where R(x) is an elastic-plastic resistance function, such as in Fig. 2(b), with the value of $k = 1362$ sec. $^{-2}$ and $R_L = 439.6$ in. sec. $^{-2}$. The solid curve in Fig. 7 represents the analytical solution of (2) and the dashed curve is the solution by Newmark's numerical method with time intervals of 0.04 seconds. Values scaled off the analog computer solution are also shown in Fig. 7. The oscillograph records Fig. 8(a)-(d) represent the solutions of (2) and three similar equations in which the input function has different intensities. In Fig. 8(e) the response of the system is shown when subjected to about 10% of critical viscous damping.

It is considered that the accuracy of the method is of about the same order as that of normal analog computer operations. This statement is made in spite of the fact that it was found that the maximum deflection was extremely sensitive to variations in the settings for R_L. However, such sensitivity is a true characteristic of an elastic-plastic system and is shown up by the analog computer. This characteristic might be missed when other methods are used to calculate the deflections, with the result that false confidence may be placed in the accuracy of the calculations.

Evaluation of Method

Using standard components, an analog computer circuit has been developed which will produce an elastic-plastic resistance relationship. Although this circuit may appear complex, an experienced operator will be able to make the necessary connections in less than half a day. Almost every dynamic problem involving an elastic-plastic system will take longer to solve by analytical methods. Numerical methods may yield answers quicker, but are not as flexible as the computer method and therefore not so suitable for design purposes. For example, it may be desirable to study the effect of changing the stiffness, or of introducing damping (Fig. 8-e), or of varying the intensity of the applied force. All this can be done by means of the analog computer literally in a matter of minutes, after the first solution has been obtained.

If the force F(t) causing the disturbance is completely irregular, as in the case of seismic disturbances, special arrangements are usually required to reproduce such an irregular function faithfully. In the simpler cases it may be possible to make a harmonic analysis of the function and then to represent it as the sum of a number of sine curves. In other cases the use of servo-mechanisms and representation by power series may be resorted to. There are a number of methods permitting the exact generation of irregular functions. Some of these are discussed by Korn and Korn [3].

Although the use of the method has been demonstrated for a single-degree-of-freedom system, its utility is more general. If the circuit shown in Fig. 4 is thought of simply as an operational unit which will

generate R(x), then its application to multi-degree-of-freedom systems becomes immediately evident. Since the R(x) generating circuit requires seven extra amplifiers, the use of two or more such circuits will normally be governed by the availability of equipment. In most cases it will not be necessary to use more than one elastic-plastic circuit in the analysis of a system. A completely elastic circuit can be used initially, to investigate which spring of the multi-degree system will enter the plastic range first. Thereafter only the circuit representing this particular spring need be of the elastic-plastic type, unless the subsequent yielding of other springs is of interest also. On this basis the method has in fact been successfully applied to a multi-degree-of-freedom system.

The circuit, in its present stage of development, functions only so long as x does not reach the plastic range in the negative direction. Work is in progress to perfect the circuit so that it will generate the elastic-plastic resistance function in both directions. One immediate application of such an improved circuit would be the investigation of the effect of yielding on the phenomenon of resonance. This would be of particular interest in the study of the effect of earthquakes on structures.

ACKNOWLEDGMENTS

This unsponsored study was carried out at the University of Michigan in the Department of Civil Engineering, of which Professor Earnest Boyce is chairman. The authors wish to express their gratitude to Professor Leo M. Legatski whose efforts are responsible for availability of an analog computer in the department.

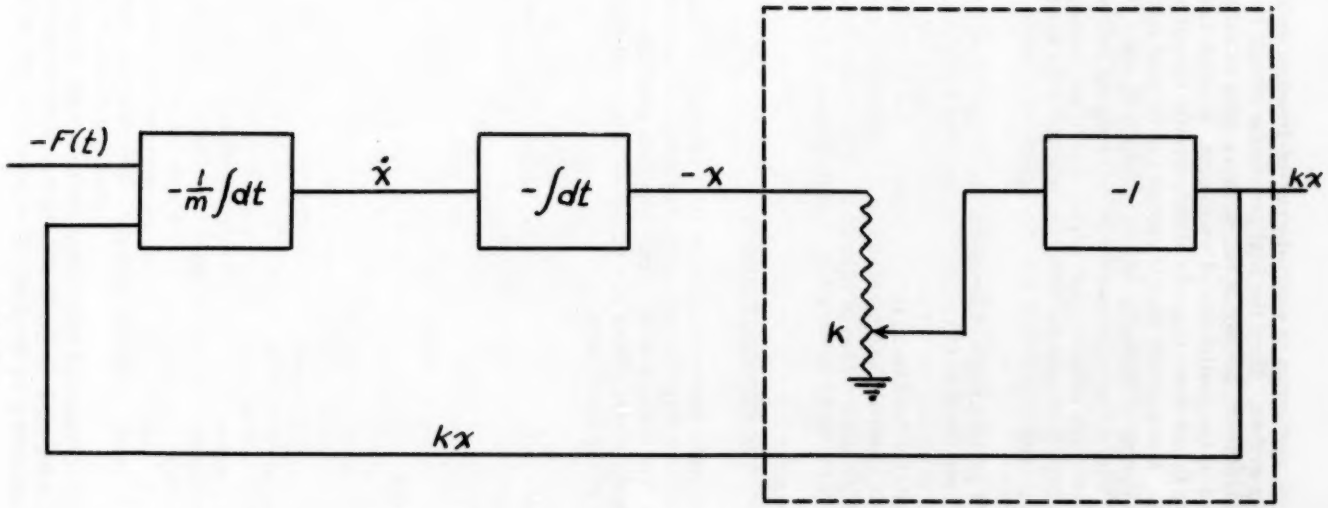

Fig. 1. Analog Computer Circuit for Undamped Single-Degree-of-Freedom Elastic System.

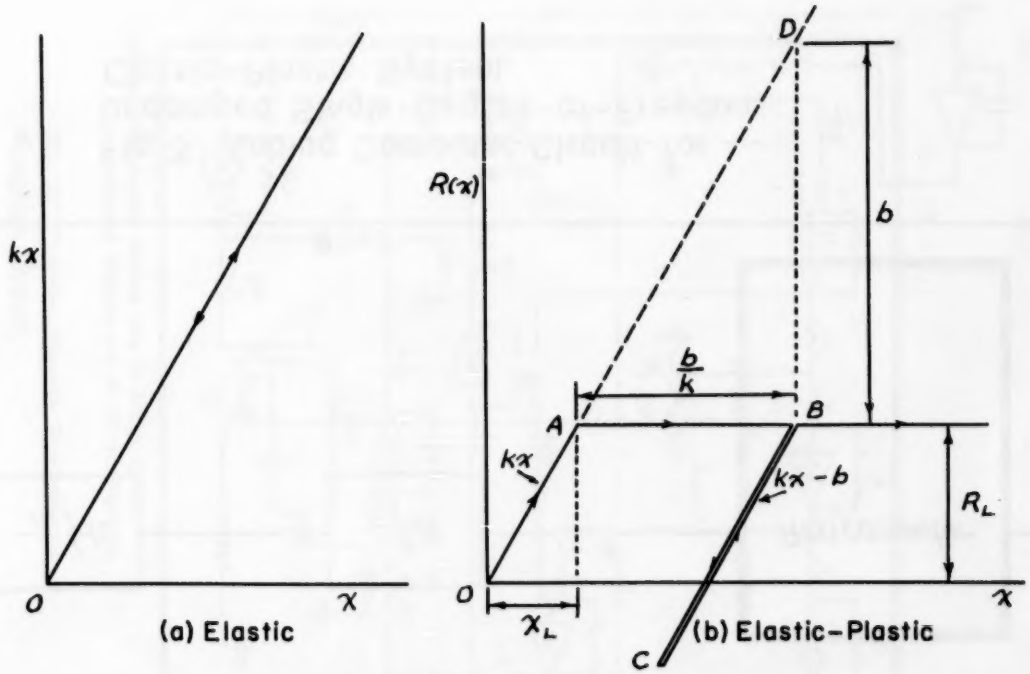

Fig. 2. Elastic and Elastic-Plastic Resistance Functions.

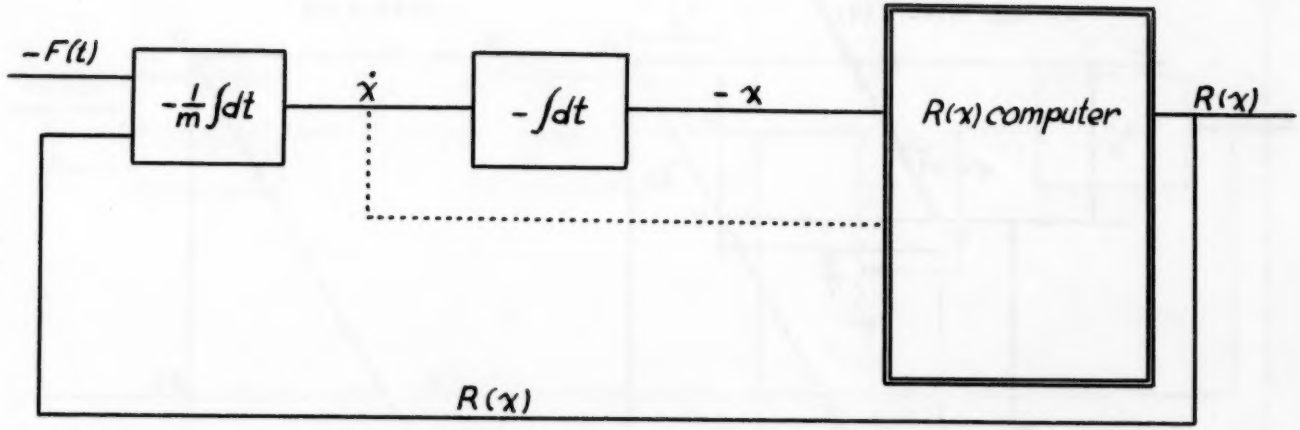

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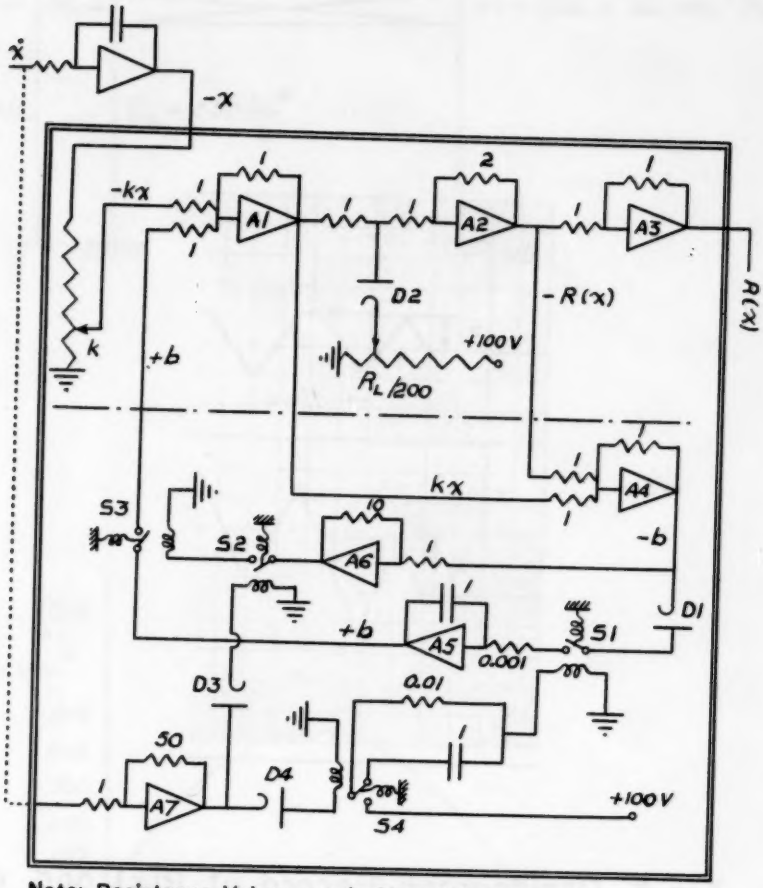
Fig. 3. Analog Computer Circuit for Undamped Single-Degree-of-Freedom Elastic-Plastic System.



Note: Resistance Values are in Mega Ohms;
Capacitance Values are in Micro Farads.

Fig. 4. Elastic-Plastic Resistance Function Circuit.

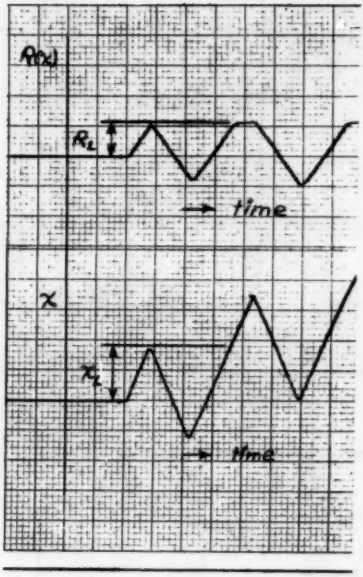
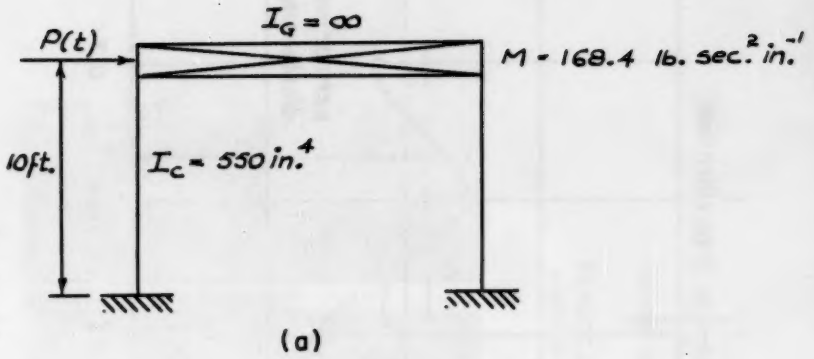
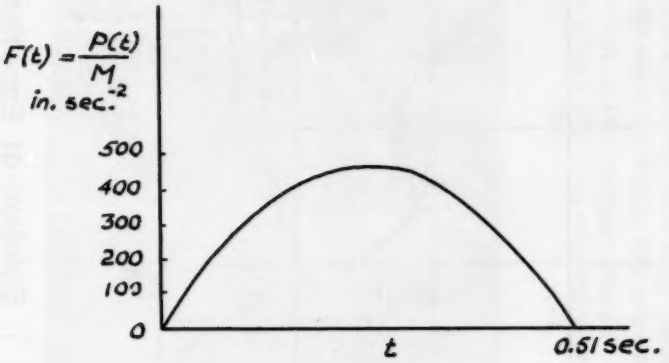


Fig. 5. Oscillograph Record of R(x) and x.



(a)



(b)

Fig. 6. Details of Single-Story Structure and Applied Force.

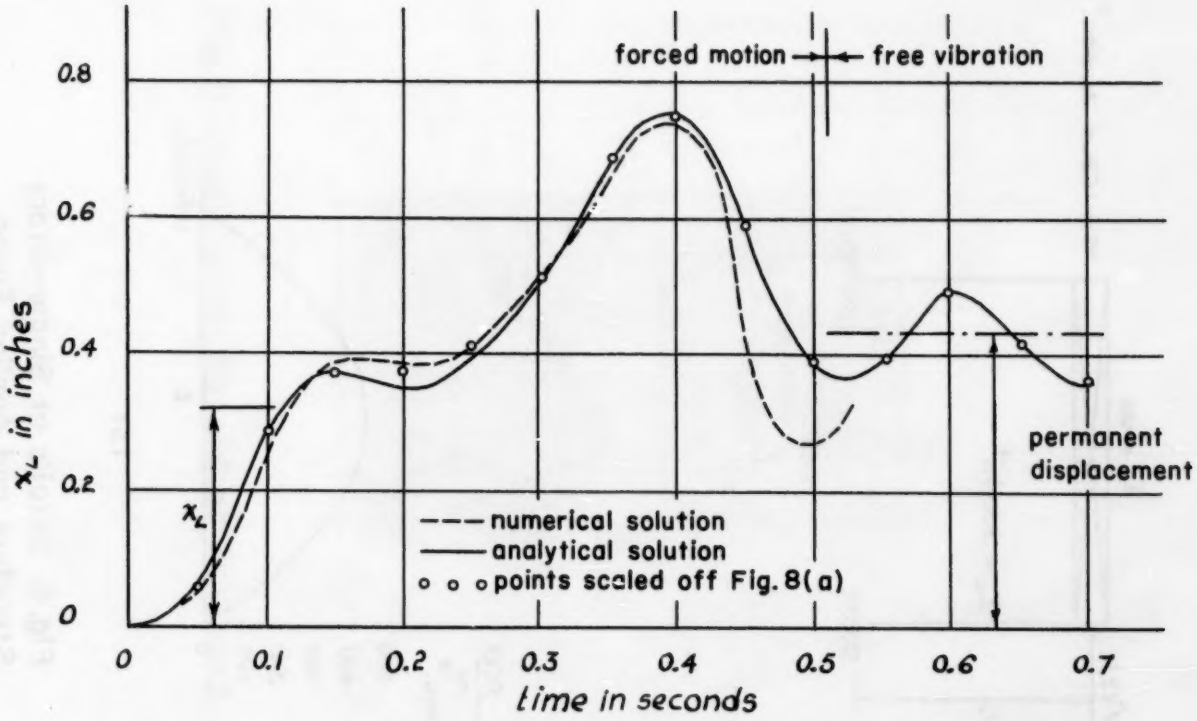
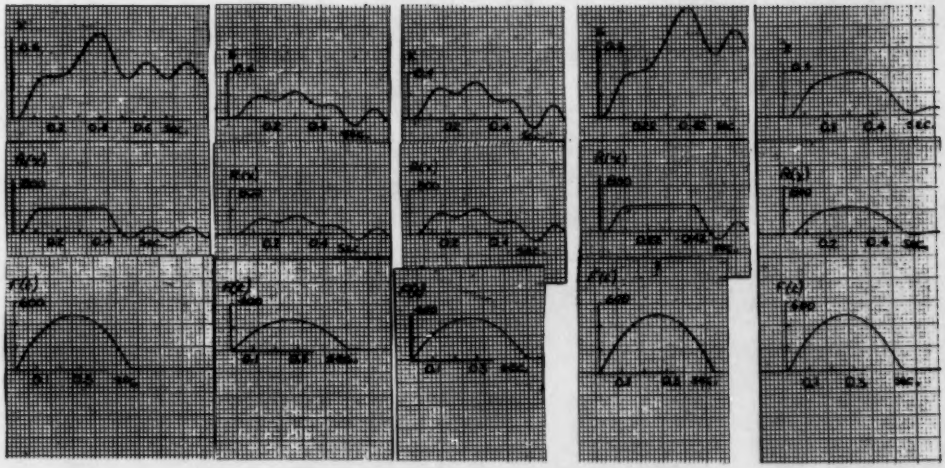


Fig. 7. Solution of Equation (2) by Various Methods.



(a) (b) (c) (d) (e)

Input Function F(t): (a) $3720t - 7300t^2$ (b) $2065t - 4060t^2$ (c) $2810t - 5520t^2$ (d) $3780t - 7420t^2$ (e) $3720t - 7300t^2$ & system has 10% critical damping.

Note: Time scale nominal. Approximate correction factor 36/37.

Fig. 8. Oscillograph Records of the Response of the Elastic–Plastic System to Various Forcing Functions.

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